Numerical Integration

Suppose a marathon runner tracks his speed (accurate to 0.5 m/s) at 10 second intervals with a GPS device and the data is saved in a table. There are N measurements in all:

measurement	1	2	3	4	5	6	7	…	i	…	N
Speed v (m/s)	2	2.5	3	3.5	3	4	4.5	…	v_i	…	v_N
Time t of interval (seconds)	0	10	20	30	40	50	60	…	t_i	…	t_N

Since distance equals speed*time, and making the assumption that each speed represents the average over the interval in which it is measured, then the total distance travelled over all the N intervals is:

$$S_N - S_1 = 2*10 + 2.5*10 + 3*10 + 3.5*10 + 3*10 + 4*10 + 4.5*10 + ... + v_i*10 + ... + v_N*10$$

Defining Δt = time step (= 10 seconds) and v_i as the average speed in the interval i, we get the general expression

$$S_N - S_1 = \sum_{i=1}^{i=N} v_i * \Delta t$$

The summation expression on the right-hand side is called the ***definite integral*** because it adds, or integrates, all the $v_i*\Delta t$ products.

Since speed v is the derivative of distance with time, dS/dt, we can write this as

$$S_N - S_1 = \sum_{i=1}^{i=N} \frac{dS_i}{dt} * \Delta t$$

(The definite integral is the inverse of the differentiation because it starts with derivatives dS/dt and generates S, whereas differentiation starts with S and generates dS/dt.)

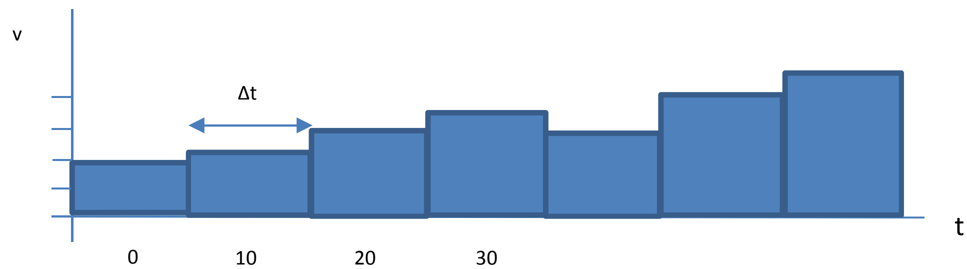
Graphical Representation of the Definite integral

When plotted as a graph of velocity against time, the definite integral

$$v_1*\Delta t + v_2*\Delta t + v_2*\Delta t + ... + v_N*\Delta t$$

is simply the sum of the shaded areas under the data.

"The definite integral is the area under the function between the start and end values"

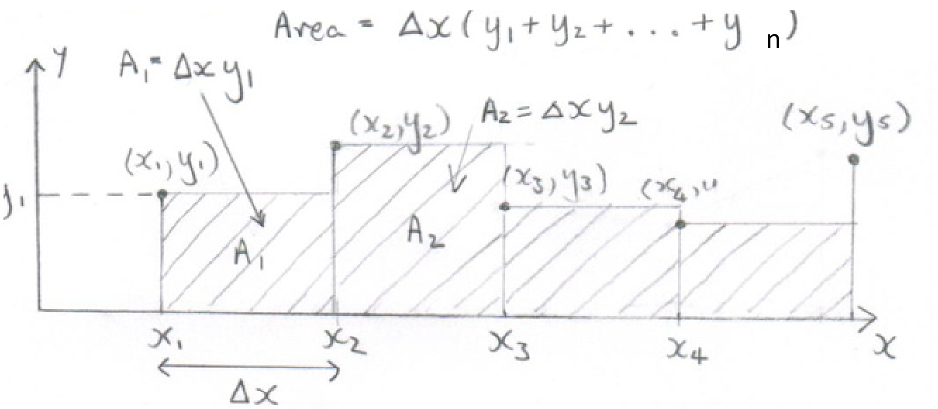


Trapezoidal Rule and Simpson's 1/3 Rule

In general we have x,y data in arrays:

y array	y_1	y_2	y_3	y_4	y_5	…	y_i	…	y_N
x array	x_1	x_2	x_3	x_4	x_5	…	x_i	…	x_N

Rules for integration are generally expressed in terms of these x,y data, and what we have found so far simply carries through. The definite integral between the first and last datum is the area under the data when plotted on a graph. In the simplest case, like the v,t data we have just looked at, the definite integral is the area under rectangles:



Trapezoidal Rule

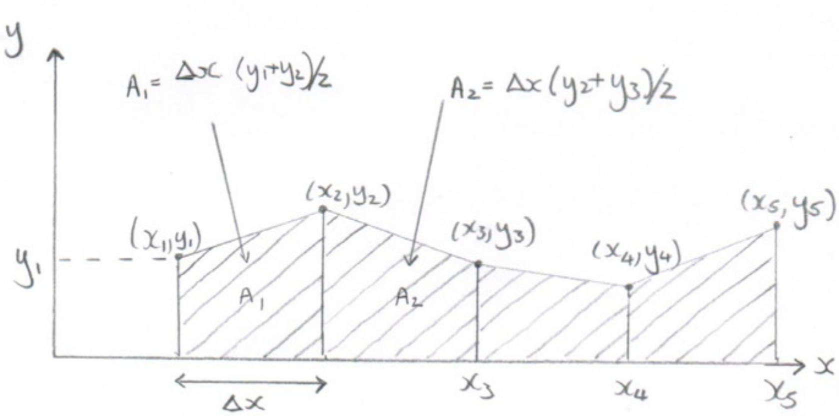
We can do better than rectangles to calculate the area – we can use trapezoids with sloping tops (this is equivalent to assuming the speed changes linearly throughout the interval rather than staying constant in the first example). Trapezoids give a more accurate measure of the area.

The area of a trapezoid is base*(average height). The base of each trapezoid is Δx and the average heights are (y1+y2)/2, (y2+y3)/2, …. giving the definite integral between the first and last data point

$$= \Delta x*(y_1+y_2)/2 + \Delta x*(y_2+y_3)/2 + \ldots + \Delta x*(y_{N-1}+y_N)/2$$

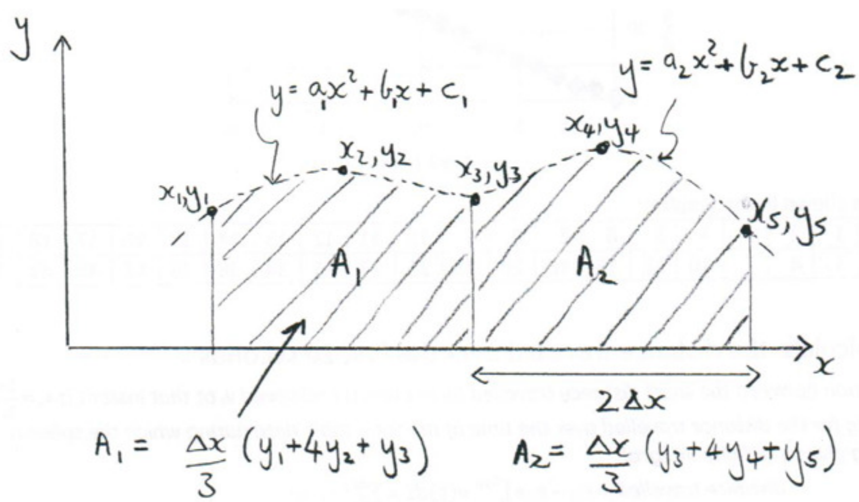
$$= \Delta x*(y_1/2 + y_2 + y_3 + y_4 + \ldots .. y_{N-1} + y_N/2)$$

This is the trapezoidal rule for the definite integral.



Simpson's 1/3 Rule

An even better calculation of the area replaces the flat tops of the trapezoids y = ax + b with the simplest curve that goes through three points the parabola $y = ax^2 + bx + c$. Now the data is taken in groups of three as shown below:



A simple calculation shows that the area under a parabola that connects the three equally spaced points (x_1,y_1), (x_2,y_2), (x_3,y_3) is

$$A = \frac{(x_2 - x_1)}{3}(y_1 + 4y_2 + y_3) = \frac{\Delta x}{3}(y_1 + 4y_2 + y_3)$$

where $\Delta x = x_2 - x_1 = x_3 - x_2$

Adding these areas together gives the definite integral for n data points

$$\frac{\Delta x}{3}(y_1 + 4y_2 + y_3) + \frac{\Delta x}{3}(y_3 + 4y_4 + y_5) + \frac{\Delta x}{3}(y_5 + 4y_6 + y_7) + \ldots.$$

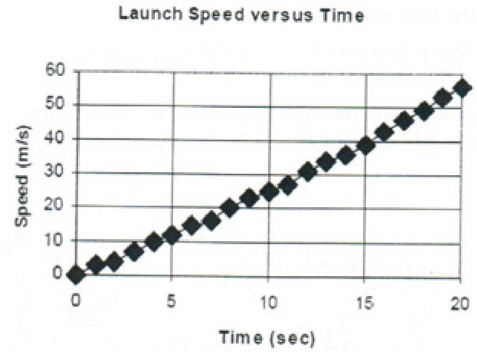
$$= \frac{\Delta x}{3}(y_1 + 4y_2 + 2y_3 + 4y_4 + 2y_5 + 4y_6 + \cdots + 4y_{n-1} + y_n)$$

This is the famous Simpson's 1/3 rule. It is the method of 1st choice for calculating the definite integral.

The first and last data points appear once and the 3rd, 5th … etc. appear twice because they are shared by adjacent parabolas.

Note that since each parabola must connect three points, the number of data points must be odd (3, 5, 7,…). If it is even, the last area can be added in as a trapezoid.

Example. The following data are from the launch of Saturn V rocket for the Apollo 11 mission on 16 july 1969 (video at **http://www.youtube.com/watch?v=F0Yd-GxJ_QM&feature=related**). The data shows the speed v (m/sec) with time (seconds) after the launch:



The data shown in the graph is:

t	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
v	0	3	4	7	10	12	15	16	20	23	25	27	31	34	36	39	43	46	49	53	56

Calculate the distance Travelled Over the First 20 Seconds

a. Using the Trapezoid Rule with $\Delta x = \Delta t = 1$ sec., and y's are the velocities

Distance travelled = $= 1.0\,(v_1/2 + v_2 + v_3 + \cdots v_{n-1} + v_n/2)$
= 0/2+3+4+7+10+12+15+16+20+23+25+27+31+34+36+39+43+46+49+53+56/2
= 521 metres

b. Using Simpson's 1/3 Rule

Distance travelled = $\frac{1}{3}(v_1 + 4v_2 + 2v_3 + 4v_4 + 2v_5 + \cdots + 2v_{n-2} + 4v_{n-1} + v_n)$
= 0.3333(0+4x3+2x4+4x7+2x10+4x12+2x15+4x16+2x20+4x23+2x25+4x27+2x31+4x34+2x36+4x39+2x43+4x46+2x49+4x53+56)
= 0.3333(0+12+8+28+20+48+30+64+40+92+50+108+62+136+72+156+86+184+98+212+56)
= 520.66 metres