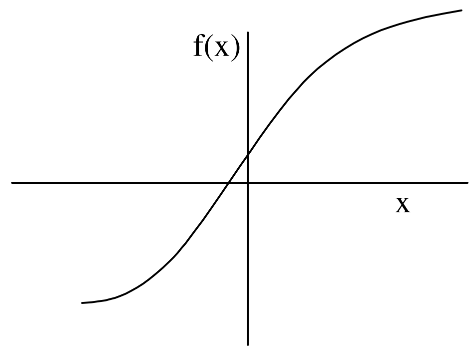
Maclaurin and Taylor Series Expansions

Truncation Errors

Often in numerical computing you want an analytic function to describe how something changes. An analytic function is a mathematical formula which is smooth, continuous and can be differentiated. It might be something like a trend of a stock price with time, or the trajectory of a missile in a game.
In this unit we look at a way of approximating analytic functions over a limited range to simple series of polynomial (powers of x) terms called **Maclaurin** and **Taylor** series.

Maclaurin series.

Very often we have a function that is interesting to us in a limited range. The simplest range would be about the origin. The function might look like:



As we narrow down the range of x at the origin, the function looks more and more like a straight line, so **very** near the origin the function is well described by:
$f(x) = A + Bx$ 1.
which is simply the equation of a straight line. This is a linear approximation.

When we include a wider range of x then it is necessary to include higher order terms that will add in the curvature. This is the basis of the series expansion in powers of x of a function over a small region:

$$f(x) = A + Bx + Cx^2 + Dx^3 + \ldots\ldots.. \qquad 2.$$

where A, B, C… are constants that can be determined from the derivatives of the function at the origin (as we will see later).
Exactly how many terms we need to include depends on the range of x and how accurately we want to approximate the function. Making that decision requires care. However, the important conclusion is that **very near** the origin, we only need a few terms, perhaps no more than:

$$f(x) = A + Bx + Cx^2 \qquad 3.$$

This is a quadratic approximation.

An essential assumption if such a series is a good approximation is that each term is smaller than the one before it. Such a series is called ***convergent*** since each successive term adds a diminishing correction and as more terms are included the series converges to an increasingly accurate approximation.

Where do the coefficients A, B, C, … come form?

Consider the full expansion:

$$f(x) = A + Bx + Cx^2 + Dx^3 + \ldots\ldots.. \qquad 1.$$

Now evaluate F(x) at x=0. All the terms in x vanish because x is 0 leaving:

$$f(0) = A$$

Now take the derivative of f at x=0:

$$f'(x) = B + 2Cx + 3Dx^2 + \ldots\ldots..$$

and evaluating it at the origin:

$$f'(0) = \frac{\partial\ F(x)}{\partial\ x}\ \Big|_{x=0} = B$$

Now continuing this process, take the second derivative of F at x = 0:

$$f''(0) = \frac{\partial^2 F(x)}{\partial\ x^2}\ \Big|_{x=0} = 2C$$

Now take the third derivative of f at x = 0

$$f'''(0) = \frac{\partial^3 F(x)}{\partial\ x^3}\ \Big|_{x=0} = 3.2.D = 3!D$$

where 3! means factorial 3.

We can continue this process as far as we want, and each derivative gives an expression for the next coefficient.

Using these expressions we can rewrite the original series in terms of the derivatives instead of the coefficients:

$$f(x) = f(0) + f'(0).x + \frac{f''(0)}{2!}.x^2 + \frac{f'''(0)}{3!}.x^3 + \ldots..$$

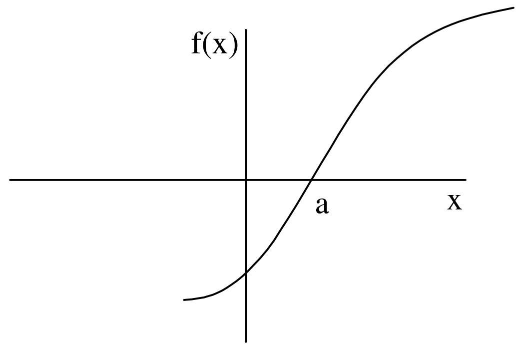
This is called a **Maclaurin** series. It can be summarized as

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0).x^k}{k!}$$

The nice feature of the series expansion is that it depends only on derivatives right at the origin. In other words, providing we know what the function is doing exactly at the origin, we can very accurately approximate its value nearby.

Taylor Series

Suppose we are interested in a series expansion in x about a point *a* which is not at the origin:



We can recognise that this is the problem we have already solved – the Maclaurin series – but with the derivatives known at x = a instead of x = 0.
In the Maclaurin series x is the distance from where we wish to evaluate the series to where we know the derivatives. In the Taylor series (x-a) is that same distance. The Taylor series becomes:

$$f(x) = f(a) + f'(a).(x-a) + \frac{f''(a)}{2!}.(x-a)^2 + \frac{f'''(a)}{3!}.(x-a)^3 + \ldots..$$

where f(a), f'(a), f''(a), ….. are the derivatives at x = a.

In compact form the Taylor series is

$$f_a(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)(x-a)^k}{k!}$$

where $f^{(k)}(a)$ is the k^{th} derivative at point a.

Setting a to 0 simply takes us back to the Maclaurin series.

The Taylor series is more general than the Maclaurin series since it uses derivatives at x = a and therefore gives more accurate results with fewer terms around x = a. The Maclaurin series uses derivatives at x = 0 and is only accurate with a small number of terms for x near 0. To use the Maclaurin series far from x = 0 needs a large number of terms and then it's better to switch to the Taylor series.

Truncation Errors

Both the Maclaurin and Taylor series have an infinite number of terms according to the definition (shown for Maclaurin):

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0).x^k}{k!}$$

but in an actual calculation only a finite number of them – the largest ones - are used. The terms that are omitted are the truncated terms. The error which results from including a finite number of terms from an infinite series is called a **truncation error**.

Suppose the series is truncated at the nth term so that only terms up to and including the n^{th} term are retained. The question is what is the error that is incurred? The answer is based on the choice of a series in the first place. By choosing a series it is implicit that each term is larger than the one that follows it so that the terms lost by truncation do not result in a large error. When each term is very much larger than the one that follows it, the series is said to be rapidly converging. If the series does not converge rapidly then it may be a poor representation of the function since many terms are needed to get an accurate result. Numerical algorithms are often classified by how rapidly they converge. To estimate the error for a converging series, the **first truncated term** will dominate and can be taken as an approximation to the error. For that reason the error is often expressed in **Big O** notation (meaning "order of") as $E \approx O(x^m)$ where m is the power of x in the first truncated term

An estimate of the truncation error for the Maclaurin series is the first truncated (first neglected) term.

Example 1

Evaluate e^x at the point the x=1 using a Maclaurin series truncated at 5 terms.

The general derivatives are:

$f(x) = e^x$; $f'(x) = e^x$; $f''(x) = e^x$; $f'''(x) = e^x$; $f''''(x) = e^x$; …..

and at the origin:

$f(0) = e^0 = 1.0$; $f'(0) = e^0 = 1.0$; $f''(0) = e^0 = 1.0$; $f'''(0) = e^0 = 1.0$; ….

So the series is

$e^x = 1 + x + x^2/2 + x^3/6 + x^4/24 + x^5/120 + \ldots\ldots.$

$e^x = 1 + x + x^2/2 + x^3/6 + x^4/24 + O(x^5)$ to 5 terms

The 5-term series at x = 1 is

$e^x = 1 + 1 + 1/2 + 1/6 + 1/24 = 65/24 = 2.708333333$

The exact value of e (to 9 decimal places) is 2.718281828 so the error is

Error = 2.718281828 - 2.708333333 = 0.009948495

The 6^{th} term, that is the first truncated term = 1/120 = - 0.008333333
So the first truncated term is a reasonable approximation to the error (84% of the exact error).

To get the ***fractional error*** (***relative error***) from the **series result alone** we write

fractional error = error/value ~ error/approximate value

~ (1^{st} truncated term)/(series approximation) = (1/120)/ 2.708333333

= 0.008333/2.708333 = 0.0031 (to 2 significant figures)

% fractional error = 0.0031 x 100 = 0.31%

The 5-term series approx is accurate to 0.31%

Example 2.
Evaluate e^{-x} at the point x=1 using a Maclaurin series truncated at 5 terms.
The general derivatives are:
$f(x) = e^{-x}$; $f'(x) = -e^{-x}$; $f''(x) = e^{-x}$; $f'''(x) = -e^{-x}$; $f''''(x) = e^{-x}$; …..
and at the origin:
$f(0) = e^{-0} = 1.0$; $f'(0) = -e^{-0} = -1.0$; $f''(0) = e^{-0} = 1.0$; $f'''(0) = -e^{-0} = -1.0$; ….

$$e^{-x} = 1 - x + x^2/2 - x^3/6 + x^4/24 - x^5/120 + \ldots\ldots.$$

$$e^{-x} = 1 - x + x^2/2 - x^3/6 + x^4/24 + O(x^5) \quad \text{to 5 terms}$$

The 5-term series at x = 1 is $e^{-x} = 1 - 1 + 1/2 - 1/6 + 1/24 = 9/24 = 0.375$
The 6^{th} term that is the first truncated term = -1/120 = - 0.008333333

(The exact value of e^{-1} is 1/e = 1/(2.718281828) = 0.367879441)

To get the fractional error from the series result alone we write

fractional error = error/value ~ error/approximate value

= (1^{st} truncated term)/(series approximation) = (1/120)/0.375

= 0.008333/0.375 = 0.0222 (3 significant figures)

% fractional error = 0.0222 x 100 = 2.22%

Example 3

Evaluate sin(x) at the point the x=0.5 using a MacLaurin series truncated after the term in x^3.

The general derivatives are:

f(x) = sin(x); f'(x) = cos(x); f''(x) = -sin(x);

f'''(x) = -cos(x); f''''(x) = sin(x); f'''''(x) = cos(x); …..

and at the origin:

f(0) = sin(0) = 0.0; f'(0) = cos(x) = 1.0; f''(0) = -sin(0) = 0.0;

f'''(0) = -cos(0) = -1.0; f''''(0) = sin(0) = 0.0; f''''5(0) = cos(0) = 1.0;

sin(x) = 0.0 + x + 0.0(x^2/2) - x^3/6 + 0.0(x^4/24) + x^5/120 up to the term in x^5
and since terms that give zero don't count, finally

sin(x) = x - x^3/6 + x^5/120 + …..

Terminating the series after the term in x^3 gives
sin(x) = x - x^3/6
and at x = 0.5
sin(0.5) = 0.5 - $(0.5)^3$/6 = 0.5 – 1/48 = 23/48 = 0.479167 to 6 significant figures.

The exact value = 0.479426 so the exact error = 0.479426 - 0.479167 = 0.00026

The term in x^5 that is the first truncated term = $(0.5)^5$/120 = 0.00026042 which is therefore a good approximation to the error

To get the fractional error from the series result alone we write

fractional error = error/value ~ error/approximate value

= (1st neglected term)/(series approximation) = ($2.6042.10^{-4}$)/ 0.479167
= $5.434.10^{-4}$

% fractional error = $5.435.10^{-4}$ x 100 = 0.054%

Example 4 - Series by Substitution

Find the first five terms in the Maclaurin series of $f(x) = e^{2x}$

We already have the series for e^x
$e^x = 1 + x + x^2/2 + x^3/6 + x^4/24 + O(x^5)$ to 5 terms

so simply write this a series in z instead of x since the name of the variable is irrelevant

$e^z = 1 + z + z^2/2 + z^3/6 + z^4/24$

then substitute $z = 2x$ to get

$e^{2x} = 1 + (2x) + (2x)^2/2 + (2x)^3/6 + (2x)^4/24$

$= 1 + 2x + 2x^2 + 4x^3/3 + 2x^4/3$

Example 5 - Series by Substitution

Find the first three terms in the series for $\sin(x^2)$

We already have the series for sin(x) for three terms
$\sin(x) = x - x^3/6 + x^5/120$

so simply write this a series in z instead of x since the name of the variable is irrelevant

$\sin(z) = z - z^3/6 + z^5/120$

then substitute $z = x^2$ we get

$\sin(x^2) = x^2 - x^6/6 + x^{10}/120$

Example 6 - Series by Product

Find the Maclaurin series up to and including the term in x^3 for the function $e^x\sin(x)$.

We have two ways of doing this:

1. Use the basic definition of the series. Calculate the derivatives of $e^x\sin(x)$, evaluate them at $x = 0$ and substitute in the basic definition.
2. Use the separate series for e^x and $\sin(x)$ and multiply them.

In this example we do option 2, but in general it is a good idea to use both methods to check the same result is obtained from each.

Taking it as a product of the two series e^x and $\sin(x)$ up to and including the term in x^3 we have

$e^x\sin(x) = (1 + x + x^2/2 + x^3/6) \,.\, (x - x^3/6)$

$= 1.(x - x^3/6)$
$+ x.(x - x^3/6)$
$+ x^2/2.(x - x^3/6)$
$+ x^3/6.(x - x^3/6)$

$= x - x^3/6 + x^2 - x^4/6 + x^3/2 - x^5/12 + x^4/6 - x^6/36$

$= x + x^2 - x^3/6 + x^3/2$ including only terms up to x^3

$= x + x^2 + x^3/3$